Compositions of Multiple Control Barrier Functions Under Input Constraints

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Introduction - Control Barrier Functions

- Control Barrier Functions (CBFs) [1] are a tool for set invariance
- General formulation
  - Let $x \in \mathbb{R}^n, u \in \mathcal{U} \subset \mathbb{R}^m$ where $\mathcal{U}$ is compact
  - Control-affine system: $\dot{x} = f(x) + g(x)u$
  - A function $h : \mathbb{R}^n \to \mathbb{R}$ is a CBF if there exists a class-$\mathcal{K}$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

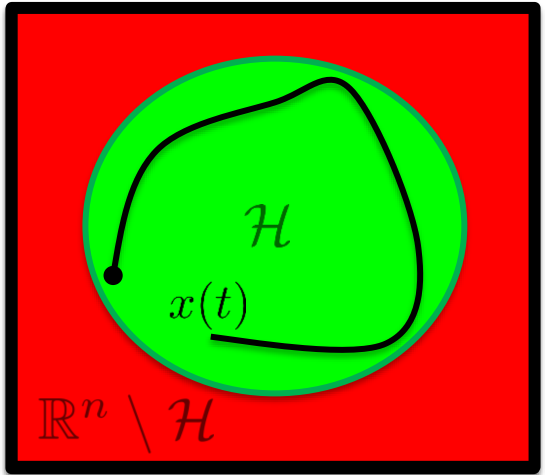
$$\inf_{u \in \mathcal{U}} \nabla h(x)(f(x) + g(x)u) \leq \alpha(-h(x))$$

for all $x \in \mathcal{H} \triangleq \{x \in \mathbb{R}^n \mid h(x) \leq 0\}$.

  - Given a CBF, the condition

$$\nabla h(x)(f(x) + g(x)u) \leq \alpha(-h(x))$$

is sufficient to render the trajectory $x(t)$ always inside $\mathcal{H}$.

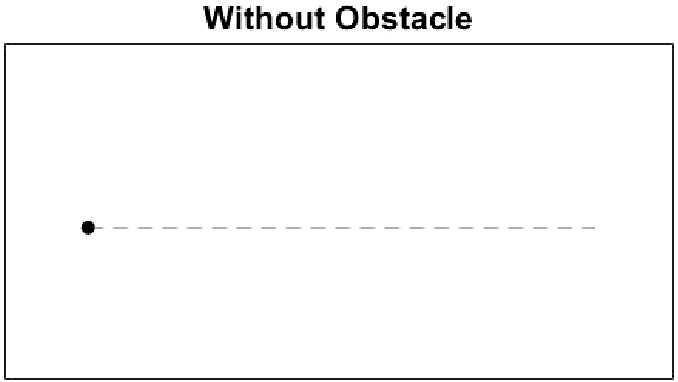


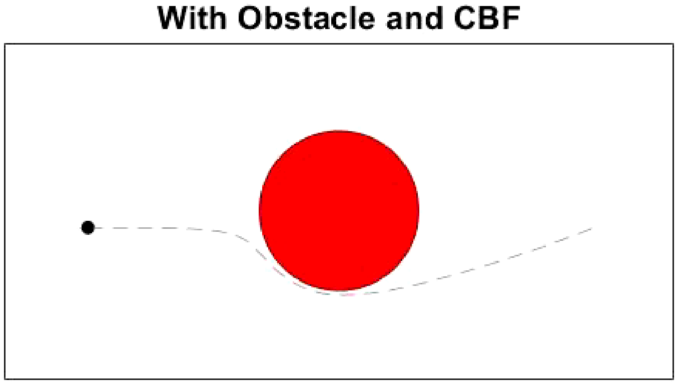
[1] Ames et al, “Control barrier functions: Theory and applications”, ECC 2019

Introduction - Control Barrier Functions

- CBFs are commonly implemented via online modifications of a nominal control law using the quadratic program

$$u = \arg\min_{u \in \mathcal{U}} \|u - u_{\text{nom}}(x)\|_2^2$$
$$\text{such that } \nabla h(x)(f(x) + g(x)u) \leq \alpha(-h(x))$$





The Problem

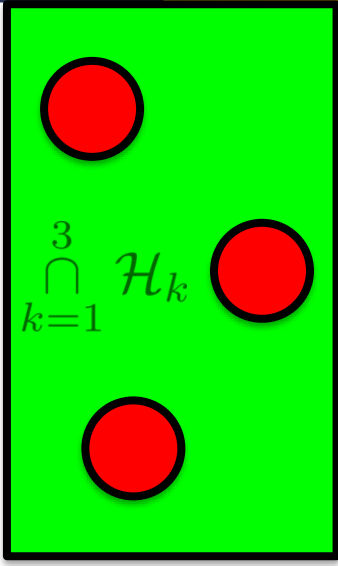
- Generally, systems operate with multiple constraints
- Multiple constraints $\{h_k\}_{k=1}^{M}$ can be handled by either
  - Developing a consolidated CBF h_c as a smooth maximum of $\{h_k\}_{k=1}^{M}$ (or other consolidation method) [1]

$$u = \arg\min_{u \in \mathcal{U}} \|u - u_{\text{nom}}(x)\|_2^2$$
$$\text{such that } \nabla h_c(x)(f(x) + g(x)u \leq \alpha(-h_c(x))$$

  - Applying multiple CBFs at once in a QP [2]

$$u = \arg\min_{u \in \mathcal{U}} \|u - u_{\text{nom}}(x)\|_2^2$$
$$\text{such that } \nabla h_k(x)(f(x) + g(x)u \leq \alpha(-h_k(x)), \forall k = 1, \cdots, N$$

- Both strategies are difficult to verify when $\mathcal{U}$ is bounded



[1] Black and Panagou, “Adaptation for validation of a consolidated control barrier function based control synthesis”, arXiv 2022
[2] Tan and Dimarogonas, “Compatibility checking of multiple control barrier functions for input constrained systems”, CDC 2022

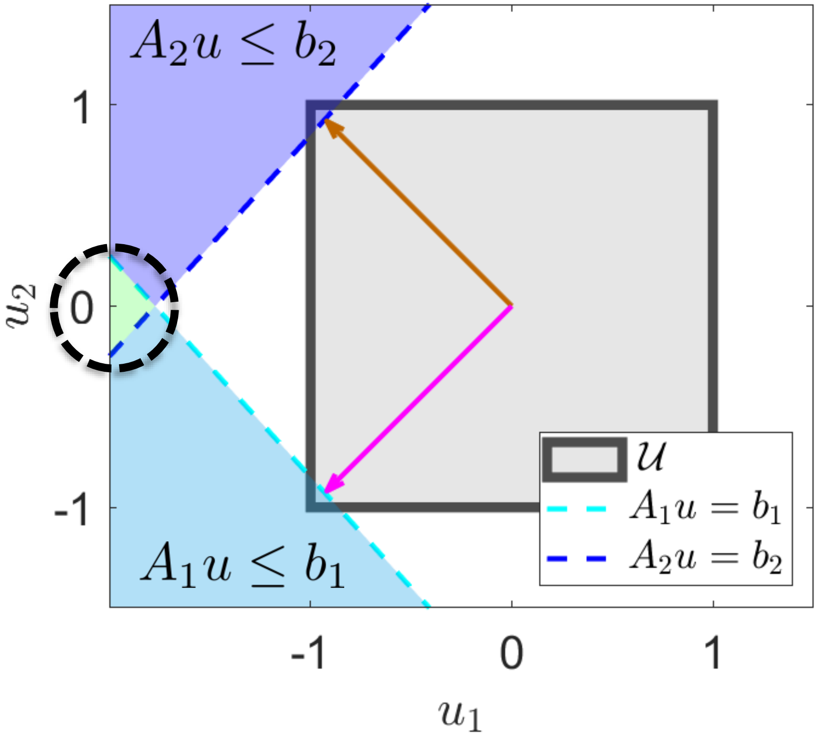
The Problem - Example

- Suppose two CBFs h_1, h_2
- Suppose a control set $\mathcal{U}$
- Suppose a state $x \in \mathcal{H}_1 \cap \mathcal{H}_2$
- This leads to two CBF conditions

$$\underbrace{\nabla h_1(x) g(x)}_{=A_1} u \leq \underbrace{\alpha_1(-h_1(x)) - \nabla h_1(x) f(x)}_{=b_1}$$

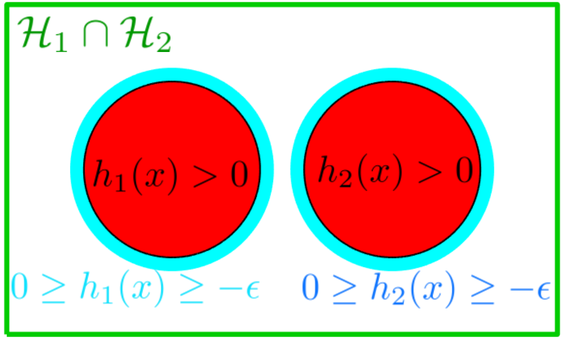
$$\underbrace{\nabla h_2(x) g(x)}_{=A_2} u \leq \underbrace{\alpha_2(-h_2(x)) - \nabla h_2(x) f(x)}_{=b_2}$$

- The conditions are individually feasible but not jointly feasible

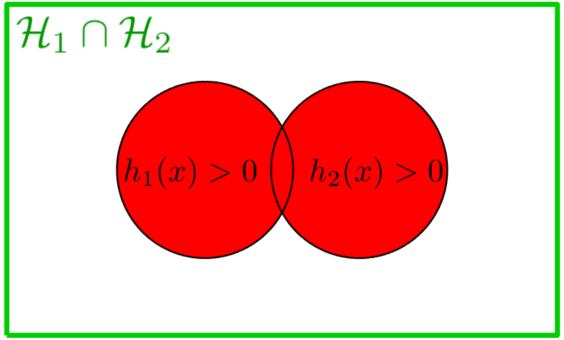


Narrower Problem

- Question 1: Do $\partial\mathcal{H}_i$ and $\partial\mathcal{H}_j$ intersect for some i, j?
  - No
    - Then treat each CBF individually in a neighborhood of its zero sublevel set [1]



  - Yes
    - Keep reading
    - This happens with most relative-degree 2 constraints



[1] Shaw Cortez, Tan, and Dimarogonas, “A robust, multiple control barrier function framework for input constrained systems”, LCSS 2022

Narrower Problem

- Question 2: Is $\cap_{k=1}^{M}\mathcal{H}_k$ a viability domain (controlled-invariant set)?
  - Yes

Proposition ([1, Thm. 1]). Let $\{h_k\}_{k=1}^{M}$ be CBFs and let $\mathcal{A} = \cap_{k=1}^{M}\mathcal{H}_k$ be a viability domain. Then the controller

$$u = \underset{u\in\mathcal{U},\delta_k\geq 1}{\arg\min} \|u - u_{\text{nom}}(t,x)\|_2^2 + \sum_{k=1}^{M} J_k\delta_k$$

$$\text{such that } \nabla h_k(x)(f(x)+g(x)u) \leq \delta_k\alpha_k(-h_k(x)),\ \forall k = 1,\cdots,M$$

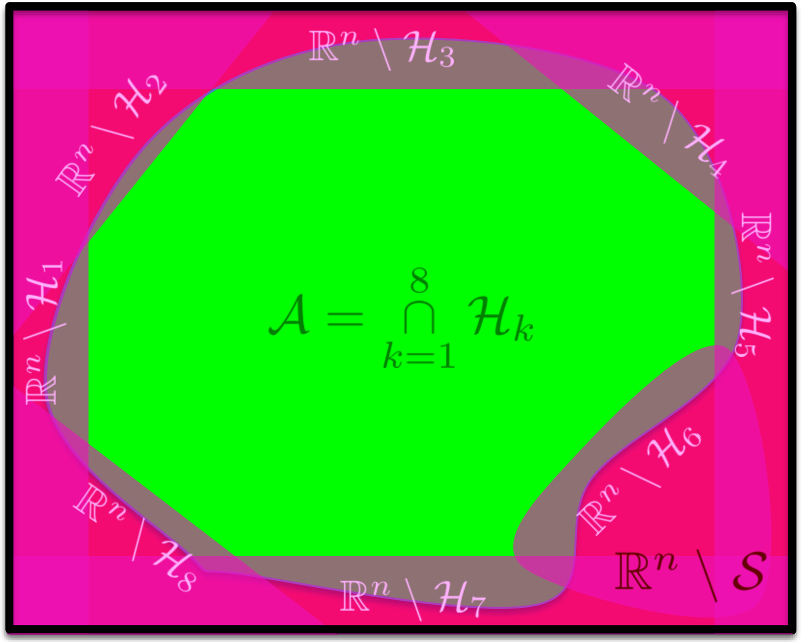
where $J_k > 0$, is feasible at every point $x \in \mathcal{A}$.

  - No
    - This paper seeks tools to modify the CBFs $\{h_k\}_{k=1}^{M}$ so as to recover a controlled-invariant set

[1] Zeng et al., "Safety-critical control using optimal-decay control barrier function with guaranteed pointwise feasibility", ACC 2021

Problem Formulation

- Goal: Find a controlled-invariant subset $\mathcal{A}$ of a specified set $\mathcal{S}$
- Tool: CBFs
  - We seek to express $\mathcal{A}$ using some number of CBFs $\mathcal{A} = \cap_{k=1}^{M} \mathcal{H}_k$ so that so we can use the QP control law on the prior slide
- Overview
  - Strategy 1 – geometry, formal guarantees
  - Strategy 2 – algorithm/heuristic



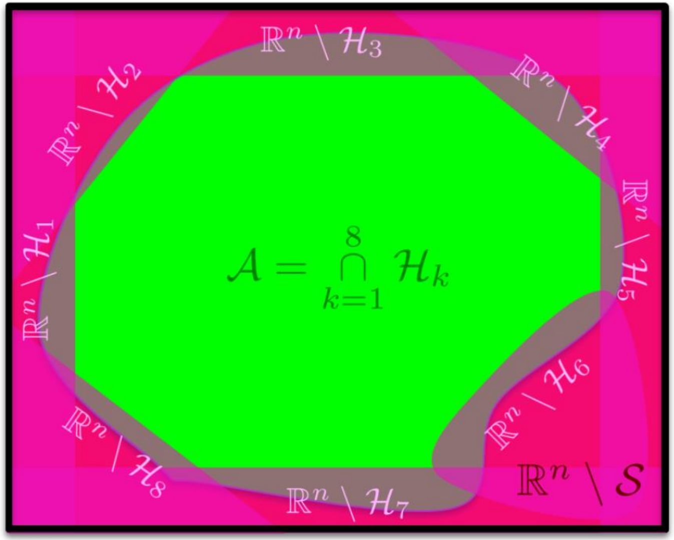
where each h_k is a CBF

Control Barrier Functions

Definition. Let $\mathcal{X} \subseteq \mathcal{S}$ and $\mathcal{Y} \subseteq \mathcal{U}$. A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a Control Barrier Function (CBF) for $(\mathcal{X}, \mathcal{Y})$ if there exists $\alpha \in \mathcal{K}$ such that

$$\inf_{u \in \mathcal{Y}} \nabla h(x)(f(x) + g(x)u) \leq \alpha(-h(x))$$

for all $x \in \mathcal{H} \cap \mathcal{X}$.



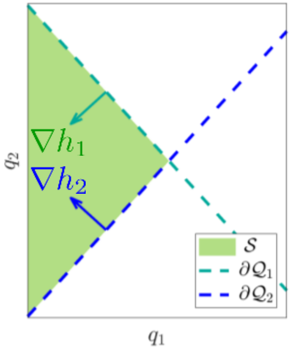
- We use $\mathcal{X}$ in place of $\mathcal{S}$ to keep track of how we will gradually restrict $\mathcal{S}$ to a smaller set $\mathcal{X}_k = \mathcal{X}_{k-1} \cap \mathcal{H}_{k-1}, \mathcal{X}_0 = \mathcal{S}$ each time that we add a CBF

Result 1: Non-interference

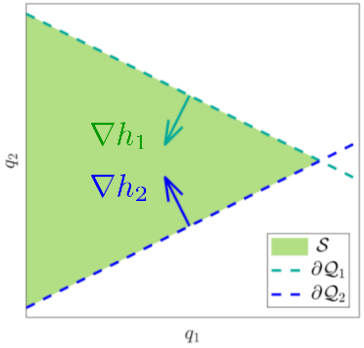
Definition. Two CBFs h_i and h_j are called non-interfering on a set $\mathcal{X} \subset \mathcal{S}$ if $(\nabla h_i(x) g(x)) \cdot (\nabla h_j(x) g(x)) \geq 0$ for all $x \in \mathcal{X}$.

A set of CBFs $\{h_k\}_{k=1}^{M}$ is call non-interfering if every pair of CBFs is non-interfering.

- Example
  - Suppose g is the identity matrix



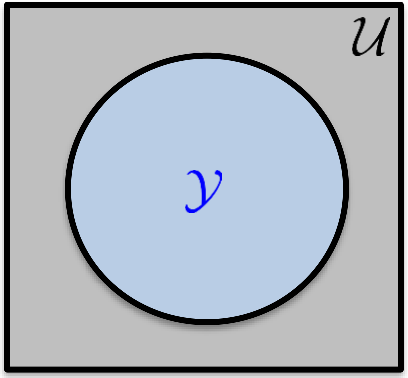
Non-Interfering

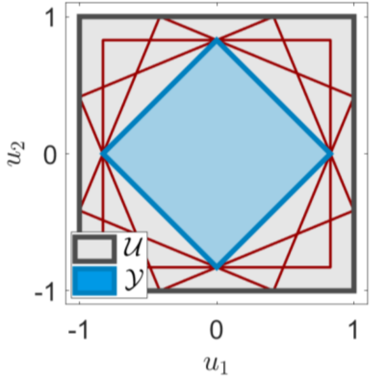


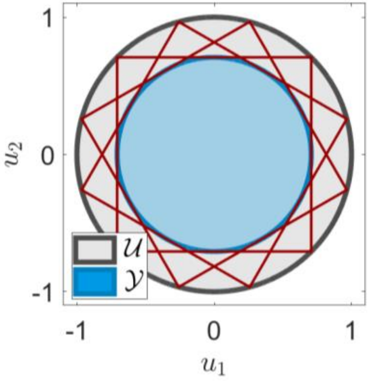
Interfering

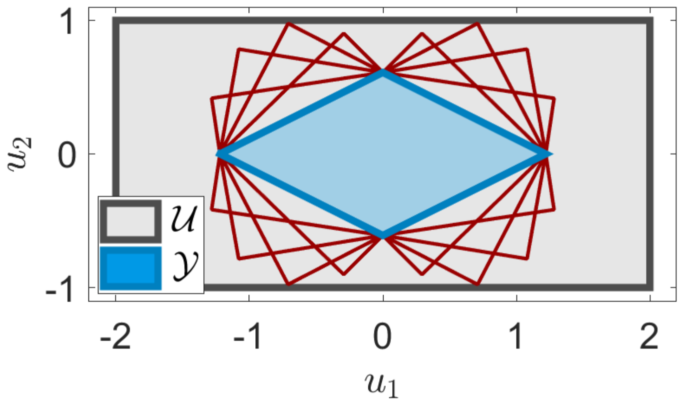
Result 1: Quadrant Extension Property

- Let $\mathcal{Y}$ be a subset of $\mathcal{U}$ that possesses the "quadrant extension property (QEP)"
  - See paper for definition and for a second similar property
- Design CBFs one-at-a-time for the smaller control set $\mathcal{Y}$ and compute the QP over all of $\mathcal{U}$









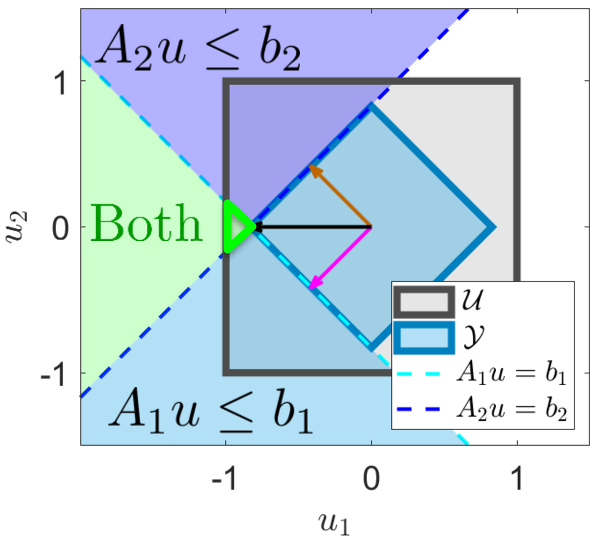
Result 1: Theorem

Theorem. Let $\mathcal{Y}$ be any set with the QEP w.r.t $\mathcal{U}$. Let $\{h_k\}_{k=1}^M$ each be a CBF for $(\mathcal{X}, \mathcal{Y})$ with $\alpha_k \in \mathcal{K}$. If $\{h_k\}_{k=1}^M$ are non-interfering, then the set

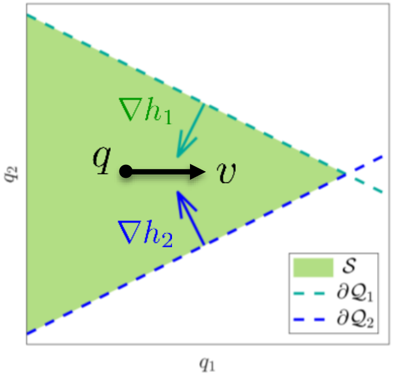
$$\boldsymbol{\mu}_{\text{all}}(x) \triangleq \{u \in \mathcal{U} \mid \nabla h_k(x)(f(x) + g(x)u) \leq \alpha_k(-h_k(x)), \forall k = 1, \cdots, M\}$$

is nonempty for all $x \in \mathcal{X} \cap (\cap_{k=1}^M \mathcal{H}_k)$.

- That is, if we design our CBFs for the smaller set of controls $\mathcal{Y}$, then the CBFs will all be feasible together over the complete set of controls $\mathcal{U}$

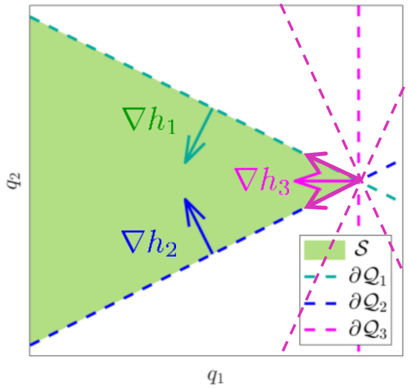


Result 2: Interfering CBFs Strategy



- Left: Position-space $q \in \mathbb{R}^2$ of a double integrator $\dot{q} = v, \dot{v} = u$
- Because the CBFs are interfering, they may allow an agent to gain excessive velocity in the direction of their intersection

- Strategy: Add a CBF



- We can solve this by adding an additional CBF to limit the agent velocity in that direction

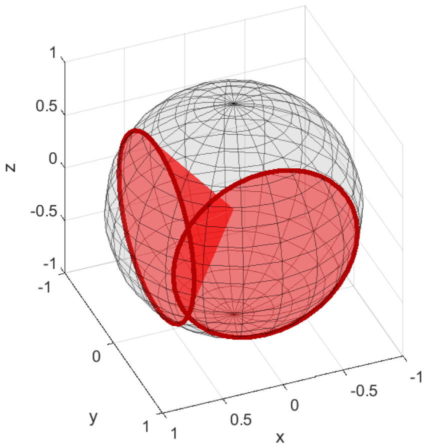
Result 2: Interfering CBFs Example

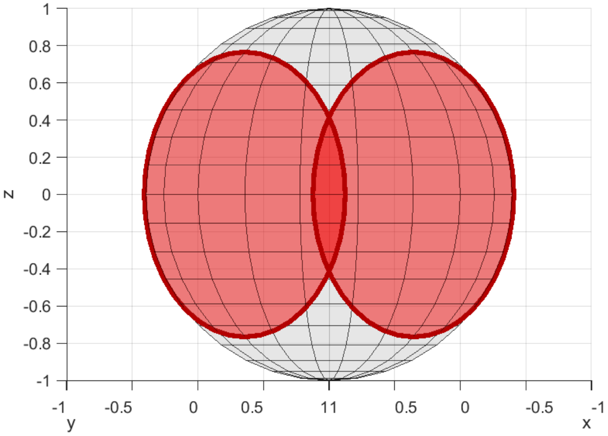
- See paper for the complete algorithm for adding CBFs
- Example:
  - 3D orientation space with two constraints

$$x = \begin{bmatrix} q \\ \omega \end{bmatrix} \in \mathbb{R}^7$$

$$\dot{q} = \frac{1}{2} \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ \omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} q$$

$$\dot{\omega} = u$$





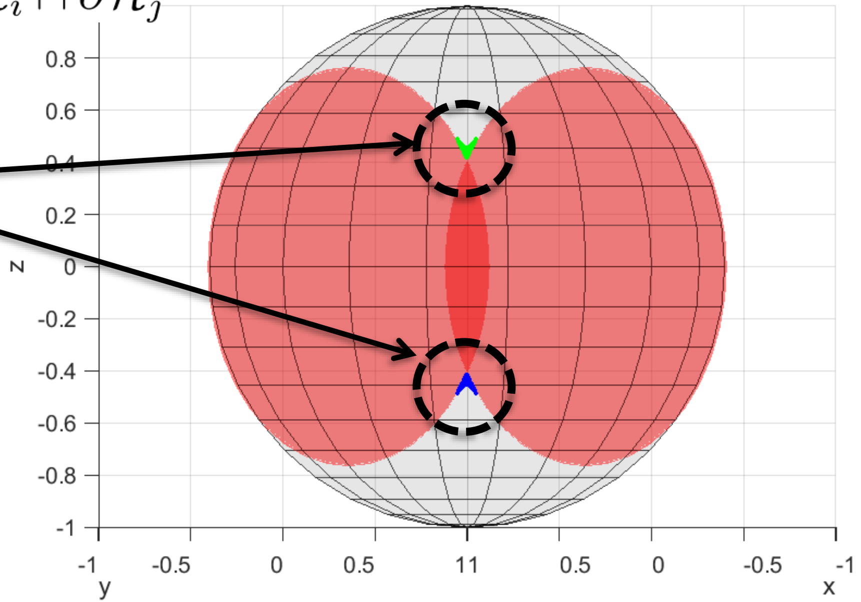
  - Red cones are unsafe states, $\mathcal{S}$ is the rest of the gray sphere
  - These two constraints intersect at a "sharp" angle and therefore are interfering

Result 2: Interfering CBFs Example

Alg-1) Identify points of conflict

– We only need to look for points
  1. In the current working set $\mathcal{X} = \mathcal{S} \cap \left(\cap_{k=1}^{M} \mathcal{H}_k \right)$
  2. In the boundary of at least two sets $\partial\mathcal{H}_i \cap \partial\mathcal{H}_j$
  3. Where the CBFs h_i, h_j are interfering
     $$(\nabla h_i(x) g(x)) \cdot (\nabla h_j(x) g(x)) < 0$$

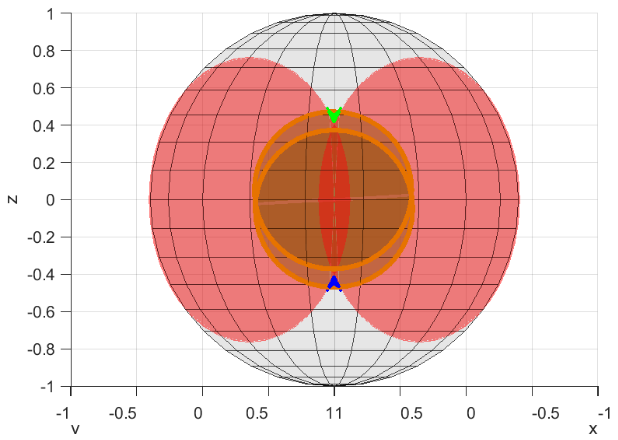
– There are two clusters of points in $\mathcal{X}$ where conflicts occur



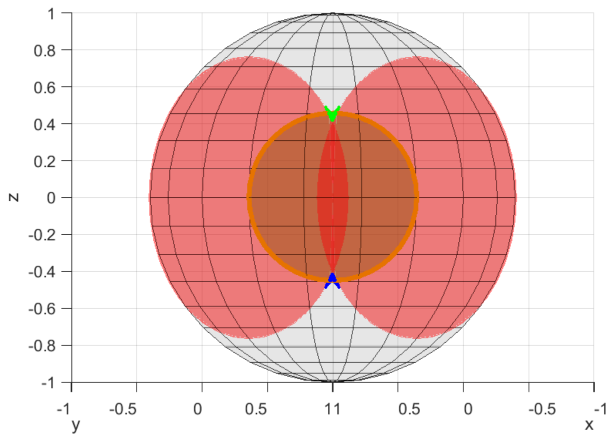
Result 2: Interfering CBFs Example

Alg-2) Remove clusters using additional CBFs

- This requires a method to produce CBFs, which will be problem-specific



Automated solution: exclude the two new brown cones



Expert solution: add one larger new cone

Alg-3) Check for conflicts again (with the new CBFs) and repeat as necessary

Conclusions

- We have presented tools for the construction of controlled-invariant sets defined using intersections of CBF sets
  - We first presented conditions for a set of non-interfering CBFs to form a controlled-invariant set
  - We then sketched an algorithm to add CBFs when the initial CBFs are interfering
- This consists entirely of offline analysis to find a controlled-invariant set as opposed to online adaptation/learning approaches
- Open questions
  - How to perform similar design for systems with disturbances
  - Can one write a general form for the added CBFs instead of having tools specific to a particular system (e.g. the cones in the presented example)
  - How to guarantee convergence of the algorithm

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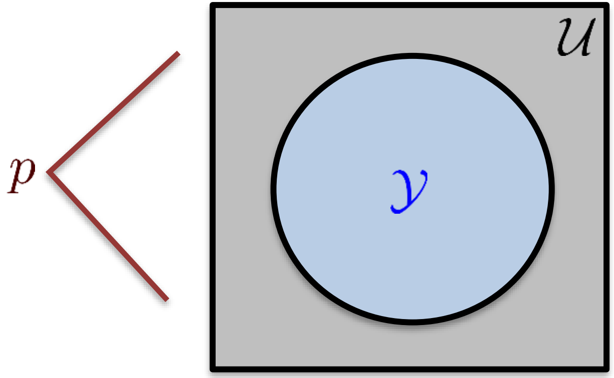
Backup - Result 1: Quadrant Extension Property

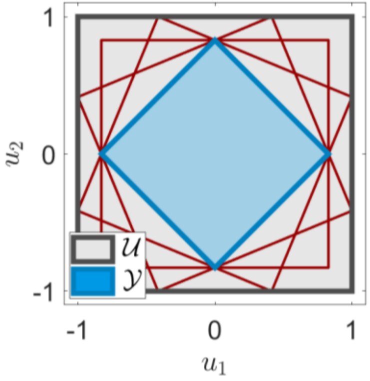
- Given a set of controls $\mathcal{Y} \subset \mathcal{U} \subset \mathbb{R}^m$
- Draw m orthogonal hyperplanes that meet at a point $p \in \mathbb{R}^m$
- Require that every hyperplane contain at least one point in $\mathcal{Y}$

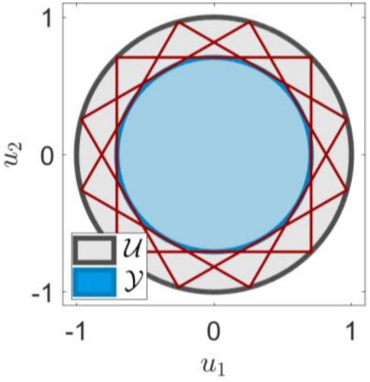


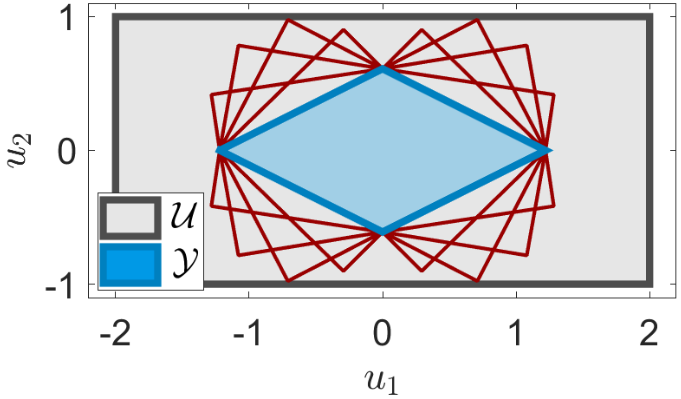
Definition. The set $\mathcal{Y} \subset \mathcal{U}$ is said to possess the quadrant extension property (QEP) w.r.t. $\mathcal{U}$ if the point p lies in $\mathcal{U}$ for any combination of m hyperplanes satisfying the above construction.

Backup - Result 1: Quadrant Extension Property

- Examples:







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